Well-defined set of exponents for a pair contact process with diffusion

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Recently it was suggested that a pair contact process with diffusion (PCPD) might represent an independent new universality class different from the directed percolation (DP) and the parity conservation (PC) class. The dynamics in the PCPD are usually controlled by two independent parameters. The critical exponents for the PCPD are known to have different values for varying values of the two independent parameters. However, once the diffusion and annihilation (or coagulation) rate in the PCPD is tuned in a way that the process without offspring production is exactly solvable, a well-defined set of the exponents for the PCPD is obtained. Then dynamics are controlled by only one independent parameter. The obtained critical exponents are different than those of DP and PC. The critical exponents satisfy the generalized hyperscaling relation within numerical errors.

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Nonequilibrium phase transitions from active (fluctuating) into inactive (absorbing) states have been studied extensively for the last twenty years [1–3]. These phase transitions into absorbing states are characterized by nonequilibrium critical behavior similar to that of an equilibrium phase transition in many respects. One usually uses the concept of scale invariance to understand nonequilibrium phase transitions as in the case of equilibrium phase transitions. One can obtain various critical exponents characterizing a certain nonequilibrium phase transition from the concept. These exponents allow us to categorize different nonequilibrium phase transitions into different universality classes. It is generally believed that nonequilibrium phase transitions into absorbing states can be categorized into a finite number of universality classes.

In reaction-diffusion processes exhibiting an absorbing phase transition, the stationary particle density ρ_s depends on a particle creation and annihilation rate. If the particle creation rate p is larger than a certain critical value p_c, ρ_s has a nonzero constant value, but if $p<p_c$, ρ_s is zero. The order parameter ρ_s vanishes algebraically as $\rho_s \sim (p-p_c)^{\beta}$ close to the critical point. Nonequilibrium phase transitions are often characterized by a spatial and temporal correlation length. The spatial and temporal correlation lengths diverge as $\xi_\perp \sim (p-p_c)^{-\nu_\perp}$ and $\xi_\parallel \sim (p-p_c)^{-\nu_\parallel}$ close to the transition point. These two correlation length scales are related by $\xi_\parallel \sim \xi_\perp^z$, where $z=\nu_\parallel/\nu_\perp$ is called the dynamic exponent. Three critical exponents $(\beta,\nu_\perp,\nu_\parallel)$ are a basic set of the critical exponents characterizing the universality class of a given reaction-diffusion process.

Among the known classes of absorbing phase transitions, the directed percolation (DP) class [2,3] is the most prominent and robust class. Continuous absorbing phase transitions into a unique absorbing state generally fall into the DP class. This class is very robust with respect to microscopic dynamic rules. Most absorbing phase transitions into many absorbing states are also known to fall into the DP class. However, when there are additional symmetries such as symmetric absorbing states, parity conservation, etc., absorbing phase transitions do not fall into the DP class.

The second well-known class is the parity-conserving (PC) class [4–6]. The PC class appears mostly in spreading processes with parity-conserving dynamics. The PC class is represented most prominently by branching-annihilating random walks with an even number of offspring, where the number of particles is preserved modulo 2. In one dimension, the parity conservation condition allows the particles to be considered as kinks between oppositely oriented domains [7,8]. From this interpretation, the parity-conserving process can be regarded as a directed percolation process with two Z_2-symmetric absorbing states [9].

Apart from these universality classes, there are a few other possible candidates [10–12]. One of them is a pair contact process with diffusion (PCPD) [11–16]. Recently it was proposed that the PCPD might represent a new class different from classes known until now. From several extensive simulations, it was confirmed that the PCPD shows new scaling properties [13,14]. Interestingly, the parity conservation condition is not relevant in this class. The most important characteristics of the PCPD is the use of a binary reaction for spreading, i.e., two particles have to meet at two adjacent places in order to create a new particle (or particles), regardless of whether the parity of the particle number is conserved or not [16]. Although many studies for the PCPD have been done for the last few years, the PCPD class remains mysterious and unclear until now. One of the factors to make the PCPD class a mysterious one is the scattered values of the critical exponents [13–16] (see Table I). Due to the scattered values, the question arises whether the PCPD represents really an independent universality class.

In this paper, we study three different models belonging to the PCPD class. When the dynamics of all the models are controlled by two independent parameters as in the models of Refs. [12–15], the critical exponents for the models have different values depending upon the value of independent parameters. However, interestingly when the diffusion and annihilation (or coagulation) rate in the PCPD is tuned in a way that the process without offspring production is exactly solvable [3], then a well-defined set of the exponents for the PCPD is obtained. In this process, the dynamics in the PCPD are controlled by only one independent parameter.

TABLE I. Estimates of the critical exponents for directed percolation, the parity-conserving class, the pair contact process with diffusion, and present models (models A, B, and C).

Class	β	$\nu_\perp$	$\nu_\parallel$
DP	0.2765	1.0969	1.734
PC	0.92(2)	1.83(3)	3.22(6)
PCPD	<0.6	1.0$\cdots$1.2	1.8$\cdots$2.1
model A	0.519(24)	1.20(9)	2.15(5)
model B	0.496(22)	1.16(6)	2.05(5)
model C	0.50(5)	1.17(7)	2.1(1)

The original PCPD process introduced by Howard and Täuber corresponds to the reaction-diffusion scheme [11]

$$\emptyset A \leftrightarrow A\emptyset \quad \text{at rate } d,$$

$$AA \rightarrow \emptyset\emptyset \quad \text{at rate } p,$$

$$AA\emptyset \rightarrow AAA \quad \text{at rate } r,$$

$$\emptyset AA \rightarrow AAA \quad \text{at rate } r,$$

where $r=(1-p)(1-d)/2$. The diffusion constant d and the pair annihilation rate p are independent parameters. The critical point p_c for the absorbing phase transition in the PCPD has a different value for each d when d is changed from 0 to 1. Then, the critical exponents characterizing the absorbing transition have different values for different p_c's [14].

Let us modify the dynamic rule of the original PCPD in order to control the diffusion and the pair annihilation rate by only a single independent parameter p. The modified model is defined on a one-dimensional lattice with L sites and periodic boundary conditions, where local variables $s_i=0,1$ indicate whether a site i is empty or occupied by a particle. The model evolves by random-sequential updates according to the following dynamic rules. For each update, a site i is randomly selected and a random number z between 0 and 1 is drawn from a flat distribution. If $p<z$ and a site i is occupied by a particle, the particle at the site i hops randomly to the left (site $i-1$) or to the right (site $i+1$). If the selected target site is already occupied, both particles at that site annihilate instantaneously. If $p>z$ and the two sites i and $i+1$ are occupied, this pair of particles generates one offspring to the left or right with equal probability. If the generated particle lands on an already occupied site, both particles at that site annihilate instantaneously. As usual, L update attempts correspond to a time increment of 1. The dynamic rules given above can also be defined in terms of the reaction-diffusion scheme (model A)

$$\emptyset A \leftrightarrow A\emptyset \quad \text{at rate } p/2,$$

$$AA \rightarrow \emptyset\emptyset \quad \text{at rate } p,$$

$$AA\emptyset \leftrightarrow AAA \quad \text{at rate } q/2,$$

$$\emptyset AA \leftrightarrow AAA \quad \text{at rate } q/2,$$

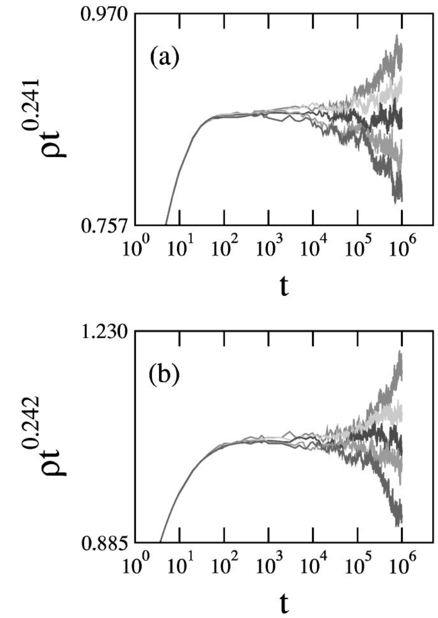

FIG. 1. (a) The figure for model A: the density of particles $\rho(t)$ times $t^{0.241}$ as a function of time for $p=0.03073$, 0.03077, 0.03081, 0.03085, and 0.03089 from top to bottom, averaged over 1000 runs on a system with 2048 sizes. (b) The figure for model B: the density of particles $\rho(t)$ times $t^{0.242}$ as a function of time for $p=0.28725$, 0.28730, 0.28735, 0.28740, and 0.28745 from top to bottom, averaged over 1000 runs on a system with 4096 sizes.

$$AAA \leftrightarrow AA\emptyset \quad \text{at rate } q/2,$$

$$AAA \leftrightarrow \emptyset AA \quad \text{at rate } q/2,$$

where $q=1-p$.

We carried out computer simulations of the modified model. We measured the density of particles $\rho(t)=(1/L)\Sigma_i s_i(t)$, initially starting with a fully occupied lattice. We found that the density decays algebraically at the critical point $p_c=0.03081(4)$ following the formula $\rho(t)\sim t^{-\delta}$ [see Fig. 1(a)]. We found the decay exponent at p_c,

$$\delta=\beta/\nu_\parallel=0.241(5). \tag{1}$$

We performed finite size simulations at the critical point to obtain the dynamic exponent $z=\nu_\parallel/\nu_\perp$. Then the density of particles should obey the following finite-size scaling form:

$$\rho(t,L)\sim t^{-\delta}f(t/L^z), \tag{2}$$

where f is a universal scaling function. Using $\delta=0.241$, the best collapse is obtained for $z=1.80(10)$ [Fig. 2(a)]. We can also determine the third independent exponent $\nu_\parallel$ from the behavior of the density below and above criticality. The density of particles follows the scaling form

$$\rho(t,\epsilon)\sim t^{-\delta}g(t\epsilon^{\nu_\parallel}), \tag{3}$$

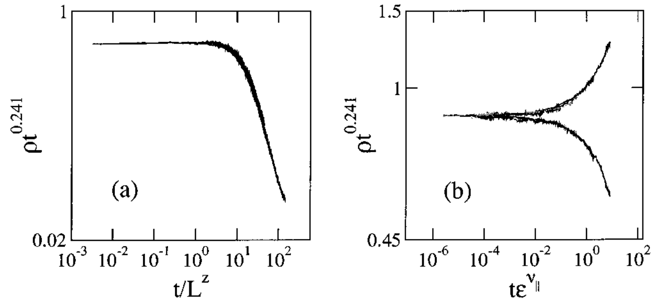

FIG. 2. The figure for model A. (a) Finite-size data collapse according to Eq. (2) for system sizes $L=180$, 256, 300, and 356 averaged over 20 000 runs. (b) Data collapse for off-critical simulations according to the scaling form (3) for $\epsilon=0.00001, 0.00002, \ldots, 0.00128$ averaged over 1000 runs.

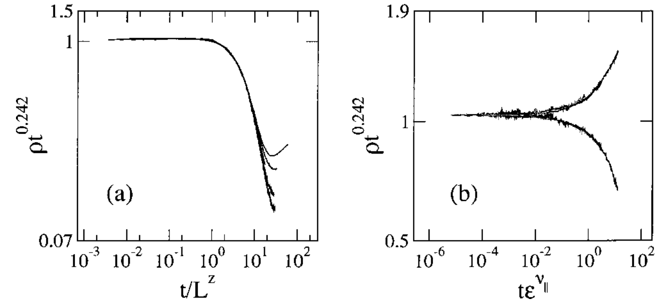

FIG. 3. The figure for model B. (a) Finite-size data collapse according to Eq. (2) for system sizes $L=64$, 90, 128, 180, and 256 averaged over 50 000 runs. (b) Data collapse for off-critical simulations according to the scaling form (3) for $\epsilon=0.00001$, 0.00002, ..., 0.00128 averaged over 1000 runs.

where $\epsilon=|p-p_c|$ denotes the distance from the critical point. By using $\delta=0.241$, the best collapse is obtained for $\nu_\parallel=2.15(5)$ [Fig. 2(b)].

Therefore, we arrive at the result

$$\beta=0.519(24), \qquad \nu_\perp=1.20(9), \qquad \nu_\parallel=2.15(5). \qquad (4)$$

We did dynamic simulations starting with a seed of a single pair of particles located on the center. We measured the survival probability $P(t)$ that the system has not yet reached an absorbing state, the average number of particles $N(t)$, and the mean square spreading from the origin $R^2(t)$ averaged over the survival runs. At criticality these quantities should obey the power laws $P(t)\sim t^{-\delta'}$, $N(t)\sim t^{\eta}$, and $R^2(t)\sim t^{2/z}$, where δ' and η are dynamical exponents. However, we found that it is very difficult to get good estimates for the critical exponents due to strong corrections to scaling. We found the values by fitting straight lines over the last decade [Fig. 4(a)],

$$\delta'=0.11(3), \qquad \eta=0.15(3), \qquad 2/z=1.04(10). \qquad (5)$$

We note that these estimates are compatible with the generalized hyperscaling relation [17]

$$d'/z=\eta+\delta+\delta' \qquad (6)$$

within numerical errors. Here d' is the spatial dimension.

Next, we considered another model, where the only difference from the first model is that the pair annihilation process $2A\rightarrow 0$ is replaced by the coagulation process $2A\rightarrow A$ in the second model. The reaction-diffusion scheme of this model (model B) is

$$\emptyset A \leftrightarrow A\emptyset \quad \text{at rate } p/2,$$

$$AA \rightarrow \emptyset A \quad \text{at rate } p/2,$$

$$AA \rightarrow A\emptyset \quad \text{at rate } p/2,$$

$$AA\emptyset \leftrightarrow AAA \quad \text{at rate } q/2,$$

$$\emptyset AA \leftrightarrow AAA \quad \text{at rate } q/2,$$

where $q=1-p$. We did computer simulations of this model. We found that the critical point is $p_c=0.28735(5)$ [see Fig. 1(b)]. We found the decay exponent at p_c,

$$\delta=\beta/\nu_\parallel=0.242(5). \qquad (7)$$

We also performed finite size simulations at the critical point to obtain the dynamic exponent $z=\nu_\parallel/\nu_\perp$. From the scaling form of the density of particles $\rho(t,L)\sim t^{-\delta}f(t/L^z)$, the best collapse is obtained for $z=1.78(5)$ by using $\delta=0.242$ [Fig. 3(a)]. We also obtained $\nu_\parallel=2.05(5)$ from the behavior of the density below and above criticality of the density of particles [Fig. 3(b)]. Therefore, we arrive at the result in the second model

$$\beta=0.496(22), \qquad \nu_\perp=1.16(6), \qquad \nu_\parallel=2.05(5). \qquad (8)$$

From the seed simulations, we obtained additional exponents [Fig. 4(b)]

$$\delta'=0.13(3), \qquad \eta=0.15(3), \qquad 2/z=1.07(6). \qquad (9)$$

In this process, the generalized hyperscaling relation is also satisfied within numerical errors.

Another known model (model C) belonging to the PCPD class in addition to the first and second model is the binary spreading process with parity conservation

$$\emptyset A \leftrightarrow A\emptyset \quad \text{at rate } p/2,$$

$$AA \rightarrow \emptyset\emptyset \quad \text{at rate } p,$$

$$AA\emptyset\emptyset \leftrightarrow AAAA \quad \text{at rate } q/2,$$

$$\emptyset\emptyset AA \leftrightarrow AAAA \quad \text{at rate } q/2,$$

$$AAA\emptyset \leftrightarrow AA\emptyset A \quad \text{at rate } q/2,$$

$$\emptyset AAA \leftrightarrow A\emptyset AA \quad \text{at rate } q/2,$$

where $q=1-p$. The model was studied recently [16]. The values of the exponents obtained from the initially fully occupied lattice are

$$\delta=0.236(10), \quad z=1.80(5), \quad \nu_\parallel=2.1(1). \qquad (10)$$

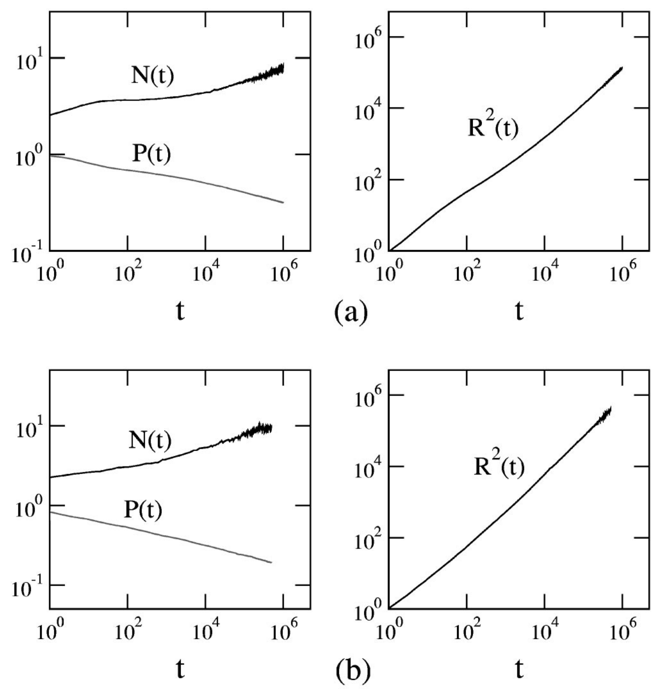

FIG. 4. The survival probability $P(t)$, the average number of particles $N(t)$, and the mean square spreading $R^2(t)$ starting with a single pair of particles for model A (a) and model B (b).

Therefore,

$$\beta = 0.50(5), \qquad \nu_\perp = 1.17(7), \qquad \nu_\parallel = 2.1(1). \qquad (11)$$

The values of the exponents from the seed simulations are

$$\delta' \approx 0.10, \qquad \eta \approx 0.20, \qquad 2/z \approx 1.15, \qquad (12)$$

without being able to estimate the errors.

The values of the critical exponents obtained from the three models are the same within the statistical errors. The critical exponents are different from those of DP and PC (see Table I). When the particle creation rate is zero in the three models, the total rate for coagulation or annihilation in those models is always twice the diffusion rate. Then the models can be solved exactly and show the same scaling behavior, i.e., $\rho(t) \sim t^{-1/2}$ in one dimension [3]. When $p > p_c$ in the three models we studied, the density of a particle decays algebraically in the long time limit according to $\rho(t) \sim t^{-1/2}$.

We studied the three models when the rate of coagulation and annihilation is three times the diffusion rate. We also found that the three models give almost the same exponents as we obtained here. But in the case of model A, it is difficult to get good estimates for the critical exponents due to strong corrections to scaling.

In conclusion, we have studied three models belonging to the PCPD class. When the dynamics of the models were controlled by two independent parameters, the critical exponents have different values depending upon the value of parameters. However, when the dynamics were controlled by only one parameter, we found that the critical exponents characterizing the dynamics of all the models have the same values within the statistical errors. In these three different models for the PCPD, we tuned the two parameters for diffusion and spreading in a particular way, namely, such that the dynamic rules become particularly simple. In fact, the total rate for coagulation and annihilation is always twice the diffusion rate. These models can be solved exactly and show the same scaling behavior if there is no offspring production. The simulation results for these models say that once one tunes the diffusion and annihilation (or coagulation) rate in a way that the process without offspring production is exactly solvable, then one has a well-defined set of the critical exponents for the PCPD. Moreover, the values of the critical exponents are different from those of DP and PC. Therefore, the PCPD does not belong to the DP and PC class.

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